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Binary nonlinearization of the super AKNS system under an implicit symmetry constraint

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Abstract

For the super AKNS system, an implicit symmetry constraint between the potentials and the eigenfunctions is proposed. After introducing some new variables to explicitly express potentials, the super AKNS system is decomposed into two compatible finite-dimensional super systems (x-part and t_n-part). Furthermore, we show that the obtained super systems are integrable super Hamiltonian systems in the supersymmetry manifold $\mathbb{R}^{4N+2|2N+2}$.

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1. Introduction

In 1988, Professor Cao proposed the mono-nonlinearization method of Lax pairs for the classical integrable (1+1)-dimensional system [1]. The key of mono-nonlinearization is to find the constraint between the potentials and the eigenfunctions of the Lax system. After choosing some distinct spectral parameters and considering the constraint, the Lax system is decomposed into finite-dimensional systems whose variables can be separated, and furthermore, the obtained finite-dimensional systems are completely integrable Hamiltonian systems in the Liouville sense. Several years later, the method was extended to classical integrable (2+1)-dimensional systems [2–4]. Thereafter, the method was continued to generalize in the following two aspects. One was the binary nonlinearization method of the Lax pairs and its adjoint Lax pairs for the classical integrable systems, which was firstly proposed by Professor Ma in [5]. And the other was the higher order constraints (i.e. implicit constraints), which were widely studied in [6–8]. In other words, the method of nonlinearization was extensively studied by many researchers in the past 20 years. Followed from this, many finite-dimensional integrable Hamiltonian systems were obtained.

In recent years, several integrable super systems [9–11] and integrable supersymmetry systems [12–14] have aroused significant interest in many mathematicians and physicists, such as Darboux transformation [15, 16], Hamiltonian structures [17–19] and so on. In very recent years, nonlinearization of the super AKNS system has been studied in [20], where we considered an explicit symmetry constraint of the super AKNS system, and we proved that under the explicit constraint, the super AKNS system was a completely integrable super Hamiltonian system in the Liouville sense. Inspired by this and the implicit constraint of the classical integrable system, the question has arisen whether an implicit symmetry constraint is available for the super AKNS system. In the present paper, we shall solve this problem.

The paper is organized as follows. In the next section, we propose an implicit symmetry constraint between the potentials and the eigenfunctions of the super AKNS system. Then in section 3, under the constraint, the super AKNS system is decomposed into two compatible finite-dimensional super systems. And furthermore, we show that the obtained finite-dimensional super systems are completely integrable in the Liouville sense. Finally, some conclusions and discussions are listed in section 4.

2. An implicit symmetry constraint of the super AKNS hierarchy

In [20], we considered the binary nonlinearization of the super AKNS system under an explicit symmetry constraint, and obtained the finite-dimensional integrable super Hamiltonian system. Here we will propose an implicit symmetry constraint of the super AKNS system. Therefore, this paper can be regarded as a continuation of [20]. For simplicity, we omit the detailed derivation of the super AKNS hierarchy, which can be referred to [20]. In what follows, we shall propose an implicit symmetry constraint between the potential and the eigenfunctions. To this aim, we consider the super AKNS spectral problem

$$\phi_x = M\phi, \qquad M = \begin{pmatrix} -\lambda & q & \alpha \\ r & \lambda & \beta \\ \beta & -\alpha & 0 \end{pmatrix}, \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{1}$$

and its adjoint spectral problem

$$\psi_x = -M^{\mathrm{St}}\psi = \begin{pmatrix} \lambda & -r & \beta \\ -q & -\lambda & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix}\psi, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{2}$$

where St means supertranspose [21].

By a similar way of the counterpart in the classical system [5, 22], it is not difficult to get the variational derivative of the parameter with respect to the potential

$$\frac{\delta\lambda}{\delta U_0} = \begin{pmatrix} \frac{\delta\lambda}{\delta r} \\ \frac{\delta\lambda}{\delta q} \\ \frac{\delta\lambda}{\delta\beta} \\ \frac{\delta\lambda}{\delta\alpha} \end{pmatrix} = \begin{pmatrix} \psi_2\phi_1 \\ \psi_1\phi_2 \\ \psi_2\phi_3 - \psi_3\phi_1 \\ \psi_1\phi_3 + \psi_3\phi_2 \end{pmatrix}, \tag{3}$$

where $U_0 = (r, q, \beta, \alpha)^T$. Imposing the zero boundary conditions $\lim_{|x|\to\infty}\phi_i = \lim_{|x|\to\infty}\psi_i = 0 (i = 1, 2, 3)$, we can verify a simple characteristic property

$$L_1\frac{\delta\lambda}{\delta U_0} = \lambda\frac{\delta\lambda}{\delta U_0}, \tag{4}$$

where

$$L_1 = \begin{pmatrix} -\frac{1}{2}\partial_x + q\partial_x^{-1}r & -q\partial_x^{-1}q & \frac{1}{2}\alpha + \frac{1}{2}q\partial_x^{-1}\beta & -\frac{1}{2}q\partial_x^{-1}\alpha \\ r\partial_x^{-1}r & \frac{1}{2}\partial_x - r\partial_x^{-1}q & \frac{1}{2}r\partial_x^{-1}\beta & -\frac{1}{2}\beta - \frac{1}{2}r\partial_x^{-1}\alpha \\ 2\beta - 2\alpha\partial_x^{-1}r & 2\alpha\partial_x^{-1}q & -\partial_x - \alpha\partial_x^{-1}\beta & -q + \alpha\partial_x^{-1}\alpha \\ 2\beta\partial_x^{-1}r & 2\alpha - 2\beta\partial_x^{-1}q & r + \beta\partial_x^{-1}\beta & \partial_x - \beta\partial_x^{-1}\alpha \end{pmatrix}, \tag{5}$$

with $\partial_x = \frac{\mathrm{d}}{\mathrm{d}x}$, $\partial_x\partial_x^{-1} = \partial_x^{-1}\partial_x = 1$.

Choosing N distinct spectral parameters $\lambda_1, \ldots, \lambda_N$, the super AKNS spectral problem (1) and the adjoint spectral problem (2) become the following finite-dimensional super systems:

$$\begin{cases} \phi_{1j,x} = -\lambda_j\phi_{1j} + q\phi_{2j} + \alpha\phi_{3j}, & 1 \leqslant j \leqslant N, \\ \phi_{2j,x} = r\phi_{1j} + \lambda_j\phi_{2j} + \beta\phi_{3j}, & 1 \leqslant j \leqslant N, \\ \phi_{3j,x} = \beta\phi_{1j} - \alpha\phi_{2j}, & 1 \leqslant j \leqslant N, \\ \psi_{1j,x} = \lambda_j\psi_{1j} - r\psi_{2j} + \beta\psi_{3j}, & 1 \leqslant j \leqslant N, \\ \psi_{2j,x} = -q\psi_{1j} - \lambda_j\psi_{2j} - \alpha\psi_{3j}, & 1 \leqslant j \leqslant N, \\ \psi_{3j,x} = -\alpha\psi_{1j} - \beta\psi_{2j}, & 1 \leqslant j \leqslant N. \end{cases} \tag{6}$$

In what follows, let us consider the traditional symmetry constraints

$$\begin{pmatrix} b_{k+1} \\ c_{k+1} \\ -2\rho_{k+1} \\ 2\delta_{k+1} \end{pmatrix} = \sum_{j=1}^{N} \gamma_j \begin{pmatrix} \frac{\delta\lambda_j}{\delta r} \\ \frac{\delta\lambda_j}{\delta q} \\ \frac{\delta\lambda_j}{\delta\beta} \\ \frac{\delta\lambda_j}{\delta\alpha} \end{pmatrix}, \tag{7}$$

where $\gamma_j (1 \leqslant j \leqslant N)$ are usual constants and $k \geqslant 0$. In [20], we have chosen $k = 0$ and $\gamma_j = 1 (1 \leqslant j \leqslant N)$ in the above constraint. Thus, we obtained an explicit symmetry constraint (i.e. the potentials can be expressed by the eigenfunctions explicitly). While in this paper, we will extend our previous work and choose $k = 1$ and $\gamma_j = -\frac{1}{2}(1 \leqslant j \leqslant N)$ in equation (7). That is to say, we obtain the following implicit symmetry constraint:

$$\begin{cases} q_x = \langle\Psi_2, \Phi_1\rangle, \\ r_x = -\langle\Psi_1, \Phi_2\rangle, \\ \alpha_x = -\frac{1}{4}(\langle\Psi_2, \Phi_3\rangle - \langle\Psi_3, \Phi_1\rangle), \\ \beta_x = -\frac{1}{4}(\langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle), \end{cases} \tag{8}$$

where $\Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^T$, $\Psi_i = (\psi_{i1}, \ldots, \psi_{iN})^T$, $i = 1, 2, 3$, and $\langle\cdot, \cdot\rangle$ denotes the standard inner product in the Euclidean space R^N. Obviously, the constraint (8) is an implicit constraint. That is to say, the potentials of the finite-dimensional super systems (6) cannot be expressed by the eigenfunctions explicitly, which is different from the constraint in [20]. In order to consider the nonlinearization of the super AKNS system under the implicit symmetry constraint (8), we should take some measures.

3. Nonlinearization of the super AKNS system under an implicit symmetry constraint

Now we are in a position to discuss the nonlinearization of the super AKNS system under the implicit symmetry constraint (8). To this aim, we firstly introduce the following new variables:

$$\phi_{N+1} = q, \qquad \phi_{N+2} = 2\alpha, \qquad \psi_{N+1} = r, \qquad \psi_{N+2} = -2\beta. \tag{9}$$

Considering the new variables (9) and substituting the constraint (8) into the system (6), we obtain the following finite-dimensional super system:

$$\begin{cases} \phi_{1j,x} = -\lambda_j\phi_{1j} + \phi_{N+1}\phi_{2j} + \frac{1}{2}\phi_{N+2}\phi_{3j}, & 1 \leqslant j \leqslant N, \\ \phi_{2j,x} = \psi_{N+1}\phi_{1j} + \lambda_j\phi_{2j} - \frac{1}{2}\psi_{N+2}\phi_{3j}, & 1 \leqslant j \leqslant N, \\ \phi_{3j,x} = -\frac{1}{2}\psi_{N+2}\phi_{1j} - \frac{1}{2}\phi_{N+2}\phi_{2j}, & 1 \leqslant j \leqslant N, \\ \phi_{N+1,x} = \langle\Psi_2, \Phi_1\rangle, & \\ \phi_{N+2,x} = -\frac{1}{2}(\langle\Psi_2, \Phi_3\rangle - \langle\Psi_3, \Phi_1\rangle), & \\ \psi_{1j,x} = \lambda_j\psi_{1j} - \psi_{N+1}\psi_{2j} - \frac{1}{2}\psi_{N+2}\psi_{3j}, & 1 \leqslant j \leqslant N, \\ \psi_{2j,x} = -\phi_{N+1}\psi_{1j} - \lambda_j\psi_{2j} - \frac{1}{2}\phi_{N+2}\psi_{3j}, & 1 \leqslant j \leqslant N, \\ \psi_{3j,x} = -\frac{1}{2}\phi_{N+2}\psi_{1j} + \frac{1}{2}\psi_{N+2}\psi_{2j}, & 1 \leqslant j \leqslant N, \\ \psi_{N+1,x} = -\langle\Psi_1, \Phi_2\rangle, & \\ \psi_{N+2,x} = \frac{1}{2}(\langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle). & \end{cases} \tag{10}$$

Obviously, equation (10) can be written in the following super Hamiltonian form:

$$\begin{cases} \Phi_{1,x} = \dfrac{\partial H_1}{\partial \Psi_1}, \quad \Phi_{2,x} = \dfrac{\partial H_1}{\partial \Psi_2}, \quad \Phi_{3,x} = \dfrac{\partial H_1}{\partial \Psi_3}, \\ \phi_{N+1,x} = \dfrac{\partial H_1}{\partial \psi_{N+1}}, \quad \phi_{N+2,x} = \dfrac{\partial H_1}{\partial \psi_{N+2}}, \\ \Psi_{1,x} = -\dfrac{\partial H_1}{\partial \Phi_1}, \quad \Psi_{2,x} = -\dfrac{\partial H_1}{\partial \Phi_2}, \quad \Psi_{3,x} = \dfrac{\partial H_1}{\partial \Phi_3}, \\ \psi_{N+1,x} = -\dfrac{\partial H_1}{\partial \phi_{N+1}}, \quad \psi_{N+2,x} = \dfrac{\partial H_1}{\partial \phi_{N+2}}, \end{cases} \tag{11}$$

where the super Hamiltonian is given by

$$H_1 = -\langle\Lambda\Psi_1, \Phi_1\rangle + \langle\Lambda\Psi_2, \Phi_2\rangle + \phi_{N+1}\langle\Psi_1, \Phi_2\rangle + \psi_{N+1}\langle\Psi_2, \Phi_1\rangle \\ + \tfrac{1}{2}\phi_{N+2}(\langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle) - \tfrac{1}{2}\psi_{N+2}(\langle\Psi_2, \Phi_3\rangle - \langle\Psi_3, \Phi_1\rangle).$$

That is to say, the nonlinearized finite-dimensional super system (10) is a super Hamiltonian system.

In what follows, let us consider the temporal part of the super AKNS hierarchy

$$\phi_{t_n} = N^{(n)}\phi = (\lambda^n N)_+\phi, \tag{12}$$

with

$$(\lambda^n N)_+ = \sum_{j=0}^{n} \begin{pmatrix} a_j & b_j & \rho_j \\ c_j & -a_j & \delta_j \\ \delta_j & -\rho_j & 0 \end{pmatrix} \lambda^{n-j},$$

where the symbol '+' denotes taking the nonnegative power of λ. When considered the N distinct spectral parameter $\lambda_1, \ldots, \lambda_N$, the temporal part of the super AKNS system becomes the following super system:

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} = \begin{pmatrix} \sum_{i=0}^n a_i\lambda_j^{n-i} & \sum_{i=0}^n b_i\lambda_j^{n-i} & \sum_{i=0}^n \rho_i\lambda_j^{n-i} \\ \sum_{i=0}^n c_i\lambda_j^{n-i} & -\sum_{i=0}^n a_i\lambda_j^{n-i} & \sum_{i=0}^n \delta_i\lambda_j^{n-i} \\ \sum_{i=0}^n \delta_i\lambda_j^{n-i} & -\sum_{i=0}^n \rho_i\lambda_j^{n-i} & 0 \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \quad 1 \leqslant j \leqslant N, \tag{13}$$

whose adjoint super system is given by

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\end{pmatrix}_{t_n} = \begin{pmatrix} -\sum_{i=0}^n a_i\lambda_j^{n-i} & -\sum_{i=0}^n c_i\lambda_j^{n-i} & \sum_{i=0}^n \delta_i\lambda_j^{n-i}\\ -\sum_{i=0}^n b_i\lambda_j^{n-i} & \sum_{i=0}^n a_i\lambda_j^{n-i} & -\sum_{i=0}^n \rho_i\lambda_j^{n-i}\\ -\sum_{i=0}^n \rho_i\lambda_j^{n-i} & -\sum_{i=0}^n \delta_i\lambda_j^{n-i} & 0\end{pmatrix}\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\end{pmatrix}, \quad 1 \leqslant j \leqslant N. \tag{14}$$

For $n = 1$, the systems (13) and (14) are exactly the spatial systems (1) and (2), respectively. In particular, as for t_2-part, the nonlinearized super systems (13) and (14) become the following system:

$$\begin{cases} \phi_{1j,t_2} = \left(-\lambda_j^2 + \frac{1}{2}qr + \alpha\beta\right)\phi_{1j} + \left(q\lambda_j - \frac{1}{2}q_x\right)\phi_{2j} + (\alpha\lambda_j - \alpha_x)\phi_{3j}, \quad 1 \leqslant j \leqslant N,\\ \phi_{2j,t_2} = \left(r\lambda_j + \frac{1}{2}r_x\right)\phi_{1j} + \left(\lambda_j^2 - \frac{1}{2}qr - \alpha\beta\right)\phi_{2j} + (\beta\lambda_j + \beta_x)\phi_{3j}, \quad 1 \leqslant j \leqslant N,\\ \phi_{3j,t_2} = (\beta\lambda_j + \beta_x)\phi_{1j} + (-\alpha\lambda_j + \alpha_x)\phi_{2j}, \quad 1 \leqslant j \leqslant N,\\ \psi_{1j,t_2} = \left(\lambda_j^2 - \frac{1}{2}qr - \alpha\beta\right)\psi_{1j} - \left(r\lambda_j + \frac{1}{2}r_x\right)\psi_{2j} + (\beta\lambda_j + +\beta_x)\psi_{3j}, \quad 1 \leqslant j \leqslant N,\\ \psi_{2j,t_2} = \left(-q\lambda_j + \frac{1}{2}q_x\right)\psi_{1j} + \left(-\lambda_j^2 + \frac{1}{2}qr + \alpha\beta\right)\psi_{2j} + (-\alpha\lambda_j + \alpha_x)\psi_{3j}, \quad 1 \leqslant j \leqslant N,\\ \psi_{3j,t_2} = (-\alpha\lambda_j + \alpha_x)\psi_{1j} - (\beta\lambda_j + \beta_x)\psi_{2j}, \quad 1 \leqslant j \leqslant N. \end{cases} \tag{15}$$

Considering the new variables (9) and the implicit constraint (8), the above finite-dimensional super system (15) becomes the following nonlinearized super system:

$$\begin{cases} \phi_{1j,t_2} = \left(-\lambda_j^2 + \frac{1}{2}\phi_{N+1}\psi_{N+1} - \frac{1}{4}\phi_{N+2}\psi_{N+2}\right)\phi_{1j} + \left(\phi_{N+1}\lambda_j - \frac{1}{2}\langle\Psi_2, \Phi_1\rangle\right)\phi_{2j}\\ \quad + \frac{1}{4}(2\phi_{N+2}\lambda_j + \langle\Psi_2, \Phi_3\rangle - \langle\Psi_3, \Phi_1\rangle)\phi_{3j},\\ \phi_{2j,t_2} = \left(\psi_{N+1}\lambda_j - \frac{1}{2}\langle\Psi_1, \Phi_2\rangle\right)\phi_{1j} + \left(\lambda_j^2 - \frac{1}{2}\phi_{N+1}\psi_{N+1} + \frac{1}{4}\phi_{N+2}\psi_{N+2}\right)\phi_{2j}\\ \quad - \frac{1}{4}(2\psi_{N+2}\lambda_j + \langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle)\phi_{3j},\\ \phi_{3j,t_2} = -\frac{1}{4}(2\psi_{N+2}\lambda_j + \langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle)\phi_{1j}\\ \quad - \frac{1}{4}(2\phi_{N+2}\lambda_j + \langle\Psi_2, \Phi_3\rangle - \langle\Psi_3, \Phi_1\rangle)\phi_{2j},\\ \phi_{N+1,t_2} = \frac{1}{2}\phi_{N+1}(\langle\Psi_1, \Phi_1\rangle - \langle\Psi_2, \Phi_2\rangle) + \langle\Lambda\Psi_2, \Phi_1\rangle\\ \quad + \phi_{N+1}^2\psi_{N+1} - \frac{1}{2}\phi_{N+1}\phi_{N+2}\psi_{N+2},\\ \phi_{N+2,t_2} = \frac{1}{4}\phi_{N+2}(\langle\Psi_1, \Phi_1\rangle - \langle\Psi_2, \Phi_2\rangle) + \frac{1}{2}(\langle\Lambda\Psi_3, \Phi_1\rangle - \langle\Lambda\Psi_2, \Phi_3\rangle),\\ \psi_{1j,t_2} = \left(\lambda_j^2 - \frac{1}{2}\phi_{N+1}\psi_{N+1} + \frac{1}{4}\phi_{N+2}\psi_{N+2}\right)\psi_{1j} - \left(\psi_{N+1}\lambda_j - \frac{1}{2}\langle\Psi_1, \Phi_2\rangle\right)\psi_{2j}\\ \quad - \frac{1}{4}(2\psi_{N+2}\lambda_j + \langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle)\psi_{3j},\\ \psi_{2j,t_2} = -\left(\phi_{N+1}\lambda_j - \frac{1}{2}\langle\Psi_2, \Phi_1\rangle\right)\psi_{1j} + \left(-\lambda_j^2 + \frac{1}{2}\phi_{N+1}\psi_{N+1} - \frac{1}{4}\phi_{N+2}\psi_{N+2}\right)\psi_{2j}\\ \quad - \frac{1}{4}(2\phi_{N+2}\lambda_j + \langle\Psi_2, \Phi_3\rangle - \langle\Psi_3, \Phi_1\rangle)\psi_{3j},\\ \psi_{3j,t_2} = -\frac{1}{4}(2\phi_{N+2}\lambda_j + \langle\Psi_2, \Phi_3\rangle - \langle\Psi_3, \Phi_1\rangle)\psi_{1j}\\ \quad + \frac{1}{4}(2\psi_{N+2}\lambda_j + \langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle)\psi_{2j},\\ \psi_{N+1,t_2} = -\langle\Lambda\Psi_1, \Phi_2\rangle - \frac{1}{2}\psi_{N+1}(\langle\Psi_1, \Phi_1\rangle - \langle\Psi_2, \Phi_2\rangle)\\ \quad - \phi_{N+1}\psi_{N+1}^2 + \frac{1}{2}\phi_{N+2}\psi_{N+1}\psi_{N+2},\\ \psi_{N+2,t_2} = \frac{1}{2}(\langle\Lambda\Psi_1, \Phi_3\rangle + \langle\Lambda\Psi_3, \Phi_2\rangle) - \frac{1}{4}\psi_{N+2}(\langle\Psi_1, \Phi_1\rangle\\ \quad - \langle\Psi_2, \Phi_2\rangle) - \frac{1}{2}\phi_{N+1}\psi_{N+1}\psi_{N+2}, \end{cases} \tag{16}$$

where $1 \leqslant j \leqslant N$.

It is a direct but tedious check that the nonlinearized super system (16) can be written in the following super Hamiltonian form:

$$\begin{cases} \Phi_{1,t_2} = \dfrac{\partial H_2}{\partial \Psi_1}, \qquad \Phi_{2,t_2} = \dfrac{\partial H_2}{\partial \Psi_2}, \qquad \Phi_{3,t_2} = \dfrac{\partial H_2}{\partial \Psi_3}, \\ \quad \phi_{N+1,t_2} = \dfrac{\partial H_2}{\partial \psi_{N+1}}, \qquad \phi_{N+2,t_2} = \dfrac{\partial H_2}{\partial \psi_{N+2}}, \\ \Psi_{1,t_2} = -\dfrac{\partial H_2}{\partial \Phi_1}, \qquad \Psi_{2,t_2} = -\dfrac{\partial H_2}{\partial \Phi_2}, \qquad \Psi_{3,t_2} = \dfrac{\partial H_2}{\partial \Phi_3}, \\ \quad \psi_{N+1,t_2} = -\dfrac{\partial H_2}{\partial \phi_{N+1}}, \qquad \psi_{N+2,t_2} = \dfrac{\partial H_2}{\partial \phi_{N+2}}, \end{cases} \tag{17}$$

where the super Hamiltonian is given by

$$\begin{aligned} H_2 = {} & -\langle \Lambda^2\Psi_1, \Phi_1\rangle + \langle \Lambda^2\Psi_2, \Phi_2\rangle + \phi\langle \Lambda\Psi_1, \Phi_2\rangle + \psi_{N+1}\langle \Lambda\Psi_2, \Phi_1\rangle \\ & + \tfrac{1}{2}\phi_{N+2}(\langle \Lambda\Psi_1, \Phi_3\rangle + \langle \Lambda\Psi_3, \Phi_2\rangle) - \tfrac{1}{2}\psi_{N+2}(\langle \Lambda\Psi_2, \Phi_3\rangle - \langle \Lambda\Psi_3, \Phi_1\rangle) \\ & + \tfrac{1}{4}(2\phi_{N+1}\psi_{N+1} - \phi_{N+2}\psi_{N+2})(\langle \Psi_1, \Phi_1\rangle - \langle \Psi_2, \Phi_2\rangle) - \tfrac{1}{2}\langle \Psi_2, \Phi_1\rangle\langle \Psi_1, \Phi_2\rangle \\ & + \tfrac{1}{4}(\langle \Psi_2, \Phi_3\rangle - \langle \Psi_3, \Phi_1\rangle)(\langle \Psi_1, \Phi_3\rangle + \langle \Psi_3, \Phi_2\rangle) - \tfrac{1}{2}\phi_{N+1}\phi_{N+2}\psi_{N+1}\psi_{N+2} \\ & + \tfrac{1}{2}\phi_{N+1}^2\psi_{N+1}^2. \end{aligned}$$

That is to say, as for t_2-part, the nonlinearized super system (16) is a finite-dimensional super Hamiltonian system. In what follows, we want to prove that for any $n \geqslant 2$, the super systems (13) and (14) can be nonlinearized, and furthermore, the obtained nonlinearized system is a finite-dimensional super Hamiltonian system. Therefore, making use of (4) and equation (5), we obtain the constrained $a_i, b_i, c_i, \rho_i, \delta_i (1 \leqslant i \leqslant N)$ in the systems (13) and (14). Only for differentiation, $\tilde{P}(U)$ denotes the new expression generated from $P(U)$ by the nonlinear constraint (8), i.e.

$$\begin{cases} \tilde{a}_i = -\frac{1}{4}\langle \Lambda^{i-2}\Psi_1, \Phi_1\rangle - \frac{1}{2}\langle \Lambda^{i-2}\Psi_2, \Phi_2\rangle, & i \geqslant 2, \\ \tilde{b}_i = -\frac{1}{2}\langle \Lambda^{i-2}\Psi_2, \Phi_1\rangle, & i \geqslant 2, \\ \tilde{c}_i = -\frac{1}{2}\langle \Lambda^{i-2}\Psi_1, \Phi_2\rangle, & i \geqslant 2, \\ \tilde{\rho}_i = \frac{1}{4}(\langle \Lambda^{i-2}\Psi_2, \Phi_3\rangle - \langle \Lambda^{i-2}\Psi_3, \Phi_1\rangle), & i \geqslant 2, \\ \tilde{\delta}_i = -\frac{1}{4}(\langle \Lambda^{i-2}\Psi_1, \Phi_3\rangle + \langle \Lambda^{i-2}\Psi_3, \Phi_2\rangle), & i \geqslant 2. \end{cases} \tag{18}$$

Substituting (18) into the super systems (13) and (14), we obtain the nonlinearized super system

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} = \begin{pmatrix} \sum_{i=0}^n \tilde{a}_i\lambda_j^{n-i} & \sum_{i=0}^n \tilde{b}_i\lambda_j^{n-i} & \sum_{i=0}^n \tilde{\rho}_i\lambda_j^{n-i} \\ \sum_{i=0}^n \tilde{c}_i\lambda_j^{n-i} & -\sum_{i=0}^n \tilde{a}_i\lambda_j^{n-i} & \sum_{i=0}^n \tilde{\delta}_i\lambda_j^{n-i} \\ \sum_{i=0}^n \tilde{\delta}_i\lambda_j^{n-i} & -\sum_{i=0}^n \tilde{\rho}_i\lambda_j^{n-i} & 0 \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \quad 1 \leqslant j \leqslant N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_{t_n} = \begin{pmatrix} -\sum_{i=0}^n \tilde{a}_i\lambda_j^{n-i} & -\sum_{i=0}^n \tilde{c}_i\lambda_j^{n-i} & \sum_{i=0}^n \tilde{\delta}_i\lambda_j^{n-i} \\ -\sum_{i=0}^n \tilde{b}_i\lambda_j^{n-i} & \sum_{i=0}^n \tilde{a}_i\lambda_j^{n-i} & -\sum_{i=0}^n \tilde{\rho}_i\lambda_j^{n-i} \\ -\sum_{i=0}^n \tilde{\rho}_i\lambda_j^{n-i} & -\sum_{i=0}^n \tilde{\delta}_i\lambda_j^{n-i} & 0 \end{pmatrix} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \quad 1 \leqslant j \leqslant N. \end{cases} \tag{19}$$

In what follows, we want to see that the nonlinearized super system (19) is a finite-dimensional super Hamiltonian system.

From equation (18), we know that the constrained co-adjoint representation equation $\tilde{N}_x = [\tilde{M}, \tilde{N}]$ is still satisfied, and furthermore, the equality $(\tilde{N}^2)_x = [\tilde{M}, \tilde{N}^2]$ is also satisfied. Therefore, let

$$\tilde{F} = \tfrac{1}{2} Str \tilde{N}^2 = \tilde{a}^2 + \tilde{b}\tilde{c} + 2\tilde{\rho}\tilde{\delta}.$$

It is not difficult to calculate that $\tilde{F}_x = 0$, which means that $\tilde{F}$ is a generating function of integrals of motion for the nonlinearized spatial system (10). Let $\tilde{F} = \sum_{n\geqslant 0} \tilde{F}_n \lambda^{-n}$, integrals of motion $\tilde{F}_n (n \geqslant 0)$ is given by the following formulas:

$$\begin{aligned}
&\tilde{F}_0 = 1, \qquad \tilde{F}_1 = 0,\\
&\tilde{F}_2 = \frac{1}{2}(\langle \Psi_1, \Phi_1\rangle - \langle \Psi_2, \Phi_2\rangle) + \phi_{N+1}\psi_{N+1} - \frac{1}{2}\phi_{N+2}\psi_{N+2},\\
&\tilde{F}_3 = \frac{1}{2}(\langle \Lambda\Psi_1, \Phi_1\rangle - \langle \Lambda\Psi_2, \Phi_2\rangle) - \frac{1}{2}\phi_{N+1}\langle \Psi_1, \Phi_2\rangle - \frac{1}{2}\psi_{N+1}\langle \Psi_2, \Phi_1\rangle\\
&\quad - \frac{1}{4}\phi_{N+2}(\langle \Psi_1, \Phi_3\rangle + \langle \Psi_3, \Phi_2\rangle) + \frac{1}{4}\psi_{N+2}(\langle \Psi_2, \Phi_3\rangle - \langle \Psi_3, \Phi_1\rangle),\\
&\tilde{F}_n = \sum_{i=2}^{n-1}\left[\frac{1}{16}(\langle \Lambda^{i-2}\Psi_1, \Phi_1\rangle - \langle \Lambda^{i-2}\Psi_2, \Phi_2\rangle)(\langle \Lambda^{n-i-2}\Psi_1, \Phi_1\rangle - \langle \Lambda^{n-i-2}\Psi_2, \Phi_2\rangle)\right.\\
&\quad - \frac{1}{8}(\langle \Lambda^{i-2}\Psi_2, \Phi_3\rangle - \langle \Lambda^{i-2}\Psi_3, \Phi_1\rangle)(\langle \Lambda^{n-i-2}\Psi_1, \Phi_3\rangle + \langle \Lambda^{n-i-2}\Psi_3, \Phi_2\rangle)\\
&\quad \left. + \frac{1}{4}\langle \Lambda^{i-2}\Psi_2, \Phi_1\rangle\langle \Lambda^{n-i-2}\Psi_1, \Phi_2\rangle\right] + \frac{1}{2}(\langle \Lambda^{n-2}\Psi_1, \Phi_1\rangle - \langle \Lambda^{n-2}\Psi_2, \Phi_2\rangle)\\
&\quad - \frac{1}{4}\phi_{N+2}(\langle \Lambda^{n-3}\Psi_1, \Phi_3\rangle + \langle \Lambda^{n-3}\Psi_3, \Phi_2\rangle) + \frac{1}{4}\psi_{N+2}(\langle \Lambda^{n-3}\Psi_2, \Phi_3\rangle - \langle \Lambda^{n-3}\Psi_3, \Phi_1\rangle)\\
&\quad - \frac{1}{2}\phi_{N+1}\langle \Lambda^{n-3}\Psi_1, \Phi_2\rangle - \frac{1}{2}\psi_{N+1}\langle \Lambda^{n-3}\Psi_2, \Phi_1\rangle, \qquad n \geqslant 4.
\end{aligned} \tag{20}$$

After a direct calculation, we have

$$\begin{cases}
\Phi_{1,t_n} = -2\dfrac{\partial F_{n+2}}{\partial \Psi_1}, \quad \Phi_{2,t_n} = -2\dfrac{\partial F_{n+2}}{\partial \Psi_2}, \quad \Phi_{3,t_n} = -2\dfrac{\partial F_{n+2}}{\partial \Psi_3},\\
\phi_{N+1,t_n} = -2\dfrac{\partial F_{N+2}}{\partial \psi_{N+1}}, \quad \phi_{N+2,t_n} = -2\dfrac{\partial F_{N+2}}{\partial \psi_{N+2}},\\
\Psi_{1,t_n} = 2\dfrac{\partial F_{n+2}}{\partial \Phi_1}, \quad \Psi_{2,t_n} = 2\dfrac{\partial F_{n+2}}{\partial \Phi_2}, \quad \Psi_{3,t_n} = -2\dfrac{\partial F_{n+2}}{\partial \Phi_3},\\
\psi_{N+1,t_n} = 2\dfrac{\partial F_{N+2}}{\partial \phi_{N+1}}, \quad \psi_{N+2,t_n} = -2\dfrac{\partial F_{N+2}}{\partial \phi_{N+2}},
\end{cases} \tag{21}$$

which means that the nonlinearized temporal system (19) is a super Hamiltonian system. In conclusion, for any $n(n \geqslant 1)$, the nonlinearized system (19) is a finite-dimensional, super Hamiltonian system. In what follows, we only want to prove that the nonlinearized system (19) is completely integrable in the Liouville sense.

To this aim, we choose the following Poisson bracket:

$$\begin{aligned}
\{F, G\} = &\sum_{i=1}^{3}\sum_{j=1}^{N}\left(\frac{\partial F}{\partial \phi_{ij}}\frac{\partial G}{\partial \psi_{ij}} - (-1)^{p(\phi_{ij})p(\psi_{ij})}\frac{\partial F}{\partial \psi_{ij}}\frac{\partial G}{\partial \phi_{ij}}\right)\\
&+ \sum_{j=1}^{2}\left(\frac{\partial F}{\partial \phi_{N+j}}\frac{\partial G}{\partial \psi_{N+j}} - (-1)^{p(\phi_{N+j})p(\psi_{N+j})}\frac{\partial F}{\partial \psi_{N+j}}\frac{\partial G}{\partial \phi_{N+j}}\right),
\end{aligned} \tag{22}$$

where $p(u)$ is a parity function of u, namely $p(u) = 0$ if u is an even variable and $p(u) = 1$ if u is an odd variable. It is not difficult to see that $\tilde{F}_n (n \geqslant 0)$ are also integrals of motion for equation (19), i.e.

$$\{\tilde{F}_{m+1}, \tilde{F}_{n+2}\} = -\frac{1}{2}\frac{\partial}{\partial t_n}\tilde{F}_{m+1} = 0, \qquad m, n \geqslant 0,$$

which means that $\{\tilde{F}_n\}_{n\geqslant 0}$ are in involution in pair.

With the help of the result of nonlinearization for a classical integrable system [5, 22, 23], it is natural for us to set

$$f_k = \psi_{1k}\phi_{1k} + \psi_{2k}\phi_{2k} + \psi_{3k}\phi_{3k}, \qquad 1 \leqslant k \leqslant N, \tag{23}$$

and verify that they are also integrals of motion of the constrained spatial system (10) and temporal system (19). Making use of (22), it is easy to find that (23) is in involution in pair. For the nonlinearized spatial system (10) and the nonlinearized temporal system (19), we choose $3N$+2 integrals of motion

$$f_1, \ldots, f_N, F_2, F_3, \ldots, F_{2N+3}, \tag{24}$$

whose involution has been verified. In what follows, we want to show the functional independence of (24). Similar to [20, 23, 24], $3N$+2 functions (24) are functionally independent at least over some region of the supersymmetry manifold $\mathbb{R}^{4N+2|2N+2}$.

Taking into account the preceding program, it is not difficult to draw a conclusion below.

Theorem 1. *The constrained (6N+4)-dimensioanl systems (10) and (19) are super Hamiltonian systems, whose 3N+2 integrals of motion (24) are in involution in pair and functionally independent over the supersymmetry manifold* $\mathbb{R}^{4N+2|2N+2}$.

Remark 1. The main differences between the finite-dimensional super Hamiltonian systems in the present paper and reference [20] can be summarized below.

(1) Due to the implicit constraint, we have to introduce other four coordinates in equation (9) such that the finite-dimensional super Hamiltonian system in the present paper is (6N+4) dimensional. However, the corresponding system in [20] is 6N dimensional.

(2) Correspondingly, the Poisson bracket in equation (22) is different from equation (36) in [20].

4. Conclusions and discussions

In this paper, we presented a new finite-dimensional integrable super Hamiltonian system of the super AKNS system. The difference between this paper and [20] lies in the symmetry constraint between the potentials and the eigenfunctions, which results in generating a different finite-dimensional super system. In [20], we have proposed an explicit symmetry constraint. Substituting the explicit constraint into the spatial system and the temporal system of the super AKNS system, we have obtained the nonlinearized super system, and furthermore, we have proved that the obtained nonlinearized system is completely integrable in the Liouville sense. However, in this paper, we proposed an implicit symmetry constraint (8), which made the potentials unable to be expressed by eigenfunctions explicitly. Therefore, we refer to the method of an implicit constraint for a classical integrable system, and introduce four new variables (9) to explicitly express potentials. After this, we obtained the super nonlinearized spatial system (10) and the temporal system (19), and proved that the obtained super system (10) and (19) is the super Hamiltonian system and have $3N$+2 integrals of motion, which is in involution in pair and functionally independent at least over some region of the supersymmetry manifold $\mathbb{R}^{4N+2|2N+2}$. Lastly, we would like to stress that the nonlinearization of the super AKNS system provides a new and systematic way to construct a finite-dimensional super Hamiltonian system, and there are very few examples [25] of the Hamiltonian system with fermionic variables in literatures. Additionally, to illustrate the potential applications of the finite-dimensional super Hamiltonian system obtained in previous sections, we would like to make the following two points of physics and mathematics, respectively. On the one hand, it is possible to find these systems in the finite-dimensional super physical theory in the future, for example, the super analog of the integrable Rosochatius deformation, because finite-dimensional integrable Rosochatus systems, which are important integrable structures in string theory, can be obtained [26] through the nonlinearization of the AKNS system. On the other hand, a finite-dimensional Neumann system [27, 28], which describes the motion of a particle on S^{N-1} with a quadratic potential in the N-dimensional space, is derived again from the AKNS by nonlinearization. So, it is possible to establish the corresponding super Neumann system in the supersymmetry case through the nonlinearization of the super AKNS system.

However, for the super AKNS system, we have not yet found another kind of implicit symmetry constraint, which will engender the finite-dimensional integrable super Hamiltonian system. The difficulty lies in the selection of new variables. Once we find proper new variables, the method of nonlinearization for the super AKNS system under new implicit symmetry constraint will be carried out. In addition, under implicit constraints, nonlinearization of the other super systems will be studied in our future work.

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